Quiz 4

Date: November 7, 2025

Question 1. 5 points

Give an example of two functions $f\colon X \to Y$ and $g\colon Y \to Z$ such that $g \circ f$ is injective but g is not injective.

Solution: Consider the function $f\colon \{1\} \to \{1,2\}$ given by $f(1) = 1$, and $g\colon \{1,2\} \to \{1\}$ given by $g(1) = g(2) = 1$. Then, evidently g is not injective, but $g \circ f\colon \{1\} \to \{1\}$ is given by $(g \circ f)(1) = 1$ and is evidently injective.

Rubric:

- **(3 pts)** Giving a correct example.
- **(2pts)** Coherence in explanation and notation (e.g., specifying domains and codomains).

Question 2. 10 points

Let $f\colon X \to Y$ be a function and let $A, B \subseteq Y$. Prove the following equality of subsets of X:

$$f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B).$$

Solution: Let $x \in f^{-1}(A - B)$. Then, $f(x) \in A - B$. Thus, $f(x) \in A$ and $f(x) \notin B$. This implies that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. Thus, $x \in f^{-1}(A) - f^{-1}(B)$.

Conversely, suppose that $x \in f^{-1}(A) - f^{-1}(B)$. Then, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. The first condition implies that $f(x) \in A$ and the second that $f(x) \notin B$. Thus, $f(x) \in A - B$, and so $x \in f^{-1}(A - B)$. ■

Rubric

- **(2 pts)** Correctly recalling the definitions of preimage and set difference.
- **(2pts)** Correct strategy (i.e., giving an elmeentwise proof).
- **(3 pts)** Correctly showing that the left-hand side is contained in the right-hand side.
- **(3 pts)** Correctly showing that the right-hand side is contained in the left-hand side.